Horn clause logic for subatomics

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Outline of the talk

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- Classical probability and logic (PRISM)
- Herbrand bases and H-interpretations (LHM)
- Quantum probability spaces
- Measurable functions for grounded predicates (closed)
- Extend the result to wff with single quantifier + induction
- H_{ES} space, QM observables as propositions
- Examples from quantum communication protocols

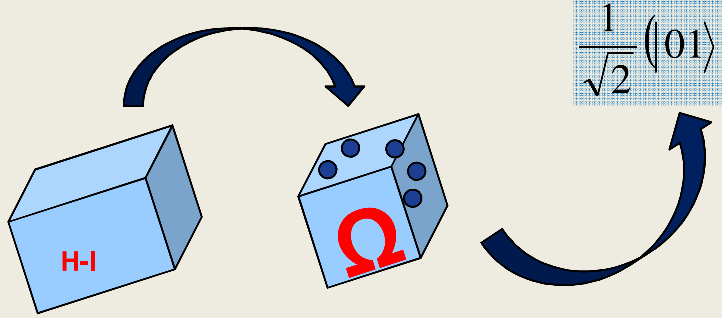
Entanglement Semantics

- Non-commutative Probability
- Quantum Probability Space
- Attach Probability Amplitudes to H-interpretations
- Projections of H
- ρ - State

$$QP = \left(\mathrm{H}^{\infty}, P(H), P\right)$$

$$\frac{1}{\sqrt{2}}\left(|01\rangle + \langle 10|\right)$$

Denotational Semantics
↕
Distribution Semantics
↕
Entanglement Semantics



Radhakrishnan Balu: Quantum probabilistic logic programming, Proc. SPIE-DSS, quantum information and computation, Baltimore MD (2015)

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First-Order Probabilistic Logic

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(3 = 5) ⇐ Pope(X)

X is Albert Einstein

X is Francis

X is DiNardo (probable)

X is Parolin (more or less probable)

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Classical Probability

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- Axiomatic theory developed by Kolmogorov
- A sample space
- σ- algebra
- Probability measure
- Product spaces, measures etc

$$CP = \left(\Omega, \sigma\left(\Omega_F\right), \rho\right)$$

Probability and Logic

Theorem (Fenstad 1967). Let Φ, φ be formulas in a language *L*. Suppose probabilities are assigned to formulas to satisfy the following:

i. P(Φ v φ) + P(Φ ∧ φ) = P(Φ) + P(Φ)
ii. P(⌐Φ) = 1 - P(Φ)
iii. P(Φ) = P(φ), if ⊢ Φ ↔ φ
iv. P(Φ) = 1, if ⊢ φ

Then there is a σ-additive probability measure λ on the set Ω of models on the Herbrand universe ***H*** for ***L***. There is also $\forall \omega \in \Omega$, a probability μω on the sets of φ[ω] such that

$$P(\phi) = \int_S \mu_\omega(\phi[\omega])d\lambda(\omega)$$

Probability measures on models
Probability measures on formulas

Sato, T., logic-based probabilistic modeling, *Proceedings of the 16th Workshop on Logic, Language, Information and Computation* (WoLLIC-2009), LNAI 5514, pp.61–71, 2009.

Distribution Semantics

- A set of probabilistic atoms F and definite clauses (Horn) R
- Probability measure on H-interpretations of F
- Extend it to DB = {F} U {R} using [1]Least Herbrand Models
- Sampling F gives atoms F' with TRUE assignments
- F' and Least Herbrand Models give TRUE atoms in {F} U {R}

[1]Doets K, "From Logic to Logic Programming," The MIT Press (1994).

$$I_j = T^j, T(I) = \{A \mid A \Leftarrow B_1 \wedge \ldots \wedge B_h (h \geq 0)\}, \{B_1 \ldots B_h\} \subseteq I$$

Minimal model that contains all the information

Distribution Semantics

- Enumerate members of Herbrand base
- Binary encode H-interpretations for enumertion
- Build a σ-algebra on the resulting Boolean lattice
- Classical Probability Space well Defined
- Attach probabilities P_F to H-interpretations of F
- Extend P_F to H-interpretations of all ground atoms
- Probability distributions with support on LHM

A	B	A ∧ B	A ∨ B
T	F	F	T
F	T	F	T
T	T	T	T
F	F	F	F

$$CP = \left(\bigcup_{j=1}^{\infty} I_j, \sigma(\Omega_F), \rho \right)$$

$$\Omega_F \overset{\Delta}{=} \prod_{i=1}^{\infty} \{0,1\}^i$$

Quantum classical probability correspondence

- Every classical r.v. can be realized as an observable
- Classical stochastic process ←→ operator process
- Non-commuting observables – main difference
- Coin Toss R.V is stochastically ~ σ^x (Pauli operator)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Quantum Probability

Hilbert space and the associated commutative von Neumann algebra (measurable functions)

$$H_{ES} = L^2\left(\Omega = \bigcup_{j=1}^{\infty} I_j, \sigma(\Omega_F), \rho \right)$$

$$< f, g > = \int_{\Omega} (fg) d\rho, f, g \in H$$

$$\mathrm{A}_{ES} = L^{\infty}\left(\Omega = \bigcup_{j=1}^{\infty} I_j, \sigma(\Omega_F), \rho \right)$$

Quantum Logic

Every closed wff is measurable:

$\Phi^{wff}(\omega) = 1$ if $\omega \models \Phi$

$= 0$ otherwise

- Hilbert space H_{ES} is separable.
- Logic is the standard one.
- Points of H_{ES} are functions with support on LHM
- Members of the algebra are observables (Φ^{wff})

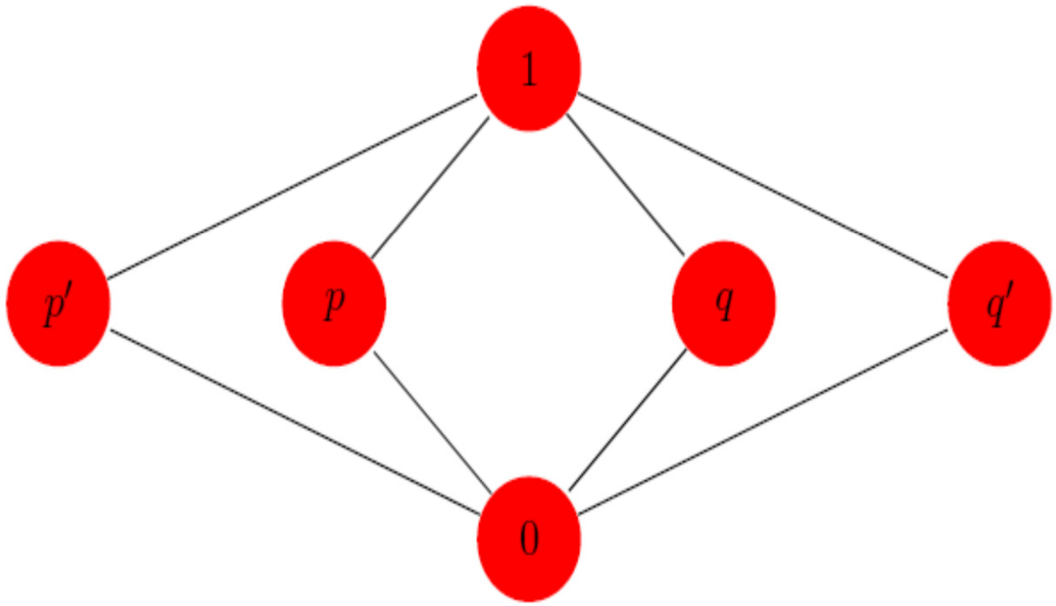
Non-Boolean lattice

$$H = C^2; e_1 = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; e_2 = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Lattice contains 6 elements
- Maximal element 1 is projection along e_1.
- Minimal element 0 is projection along e_2.
- Projections p and q are along vectors at an angle ±Θ to e_1.
- Projections p’ and q’ are orthogonal to p and q respectively

Non-Boolean Lattice

The Hasse diagram of the logic with incompatible observables resembling a ”Chinese lantern”.



We can choose angle Θ such that probabilities for p and q are 0.999999. Still, their Joint probabilities is zero as p^q = 0.

Radhakrishnan Balu: quantum probabilistic logic based computation and computing, e-Book to be published by SPIE (Fall 2015)

Gleason's theorem

Fundamental result in Quantum probability

Every probability measure can be represented as a state using unit vector (density matrix)

Theorem [1] (Gleason): The set S of probability distributions on **P(H)** is convex. If *dim* **H** ≥ 3 an element μ ε S is external if and only if there is a unit vector u such that μ(P) = <u, Pu> for all P ε **P(H).**

[1]Gleason, "Measures on the closed subspaces of Hilbert spaces.", J. Math. Mechanics,6 (1957)

Realizing Correlations in HS

Proposition[1]: Let C = ((c_{ij})) be a positive definite matrix with c_{ii} = 1 for each i,j ≤ n. Then there exist a positive integer k ≤ n, spin observables X_i, 1 ≤ i ≤ n and a pure state u in a Hilbert space of dimension k such that:

$$\langle u, X_i u\rangle = 0, \langle u, X_i X_j u\rangle = c_{ij}, 1 \le i, j \le n$$

In the case of the covariance matrix with n = 3 the usual Bell state basis spans the Hilbert space as shown below:

$$H_1 = C^2, H_2 = C^2, H = H_1 \otimes H_2$$

$$\Phi^{\pm} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B\right)$$

$$\Psi^{\pm} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B\right)$$

[1]K. R. Parthasarathy: An Introduction to Quantum Stochastic Calculus, Birkhauser, Basel (1992)

Extension to first order logic via induction

$$\mathrm{P}_{DB}\left(\varphi(t_1),\cdots,\varphi(t_n)\right)=\mathrm{P}_{DB}\left(\varphi(t_{i_1}),\cdots,\varphi(t_{i_n})\right)$$

$$\sum_{x_{n+1}}\mathrm{P}_{DB}\left(\varphi(t_1),\cdots,\varphi(t_n)=x_{n+1}\right)=\mathrm{P}_{DB}\left(\varphi(t_1),\cdots,\varphi(t_n)\right)$$

1. Tensor product of quantum states w.r.t a stabilizing sequence
2. Use Kolmogorov extension theorem then Spectral theorem

$$\{\forall,\exists\}\overset{\oplus}{\bigcup}\{\forall,\exists\}=\{\forall,\exists,\exists,\forall\}$$

Quantum Communication Protocols

Classical

r.v

quantum

module Qubit
1. state : [0..3]; // 0 is |0>, 1 is |1>, 2 is |+>, 3 is |->
2. result : [0..1]; // Result of measurement in standard basis
3. state-1 = 2 ← (state = 0), [$\sigma_X + \sigma_Z$]);
4. state-1 = 3 ← (state = 1), [$\sigma_X + \sigma_Z$]);
5. state-1 = 0 ← (state = 2), [$\sigma_X + \sigma_Z$]);
6. state-1 = 1 ← (state = 3), [$\sigma_X + \sigma_Z$]);
7. state-1 = 0, result-1 = 0 ← (state = 0), [σ_Z]);
8. state-1 = 1, result-1 = 1 ← (state = 1), [σ_Z]);
9. state-1 = 0, result-1 = 0 ← (state = 2), [σ_Z]);
10. state-1 = 1, result-1 = 1 ← (state = 2), [σ_Z]);

endmodule

$$|+\rangle\langle+| \Leftarrow |0\rangle\langle 0|, [\sigma_X + \sigma_Z];$$

$$|-\rangle\langle-| \Leftarrow |1\rangle\langle 1|, [\sigma_X + \sigma_Z];$$

$$|0\rangle\langle 0| \Leftarrow |+\rangle\langle+|, [\sigma_X + \sigma_Z];$$

$$|1\rangle\langle 1| \Leftarrow |-\rangle\langle-|, [\sigma_X + \sigma_Z];$$

$$|0\rangle\langle 0| \Leftarrow |0\rangle\langle 0|, [\sigma_Z]^*;$$

Implicit universal quantification over qudit registers

Conditional Expectation

- Commutant of A is the set of bounded linear operators of H, members commute with every element of A.

$$A' = \{C : [C, A] = 0, C \in B(H)\}$$

- A' need not be a commutative algebra

- Conditional expectations are defined w.r.t commutants

- Formal expression and the corresponding Horn clauses:

$$P[D \mid A] = \sum_i \frac{P(DA_i)}{P(A_i)} A_i ; D \in A'$$

$$P[D \mid A] \Leftarrow (D \in A') \wedge \sum_i \frac{P(DA_i)}{P(A_i)} A_i \wedge spec(A) = \{A_i\}$$

$$P[D \mid A] \Leftarrow (D \in A'), \frac{P(DA_i)}{P(A_i)} A_i, A_i \in spec(A)$$

Probe Model for Incompatibles A,B

System space, Probe space, and composite

$$\left(N = C^n \otimes N_p = C^m, P(X) = Tr\{\rho X\} \otimes P_p\right)$$

$$A = \sum_{a \in spec(A)} a P_a; \qquad \left(A \otimes I\right) \Leftrightarrow \left(U^*\left(I \otimes A'\right)U\right)$$

We are copying observable A to A' in another HS

Not violating no-cloning theorem

$$P_a' = \psi_a \psi_a^*; \qquad U = \sum_{a \in spec(A)} P_a \otimes X'_{ap};$$

$$X_{ab}' = \psi_b \psi_a^* + \psi_a \psi_b^* + \sum_{c \neq a,b} \psi_c \psi_c^*; X_{aa}' = I;$$

Horn clauses for incompatibles A,B

$$U^*(I \otimes P'_c)U = P_c \otimes P'_p + (1 - P_c) \otimes P'_c \; if\,(c \neq p).$$

$$U^*(I \otimes P'_c)U = \sum_a P_a \otimes P'_a.$$

$$\frac{(P \otimes P_p)(U^*(I \otimes P'_c)U)(P_c \otimes I)}{(P \otimes P_p)(P_c \otimes I)} = 1, \forall c.$$

$P_c \otimes P'_p$

Term cancels

The unitary interaction U copies observable A onto probe observable A'.

Horn clauses for incompatibles A,B

$$U^*(I \otimes P'_c)U = P_c \otimes P'_p + (1 - P_c) \otimes P'_c$$

$$probe\ (B, A') \Leftarrow [U^*(I \otimes A')U, U^*(B \otimes I)U] = 0;$$

$$P[U^*(B \otimes I)U \mid A] \Leftarrow probe(B, A'), \sum_i \frac{P_p(BA_i)}{P_p(A_i)} U^*(I \otimes A_i)U, spec(A) = \{A_i\};$$

Quantum state = $\left(P \otimes P_p; P_p(X) = tr\{XA_{p'}\}, p' \in spec(A')\right)$

L. M. Bouten, R. Van Handel, and M. R. James: An introduction to quantum filtering, preprint, http://arxiv.org/math.OC/0601741, (2006)

Horn clauses for incompatibles A,B, and C

$$(A \otimes I \otimes I) \Leftrightarrow (U^{*}(I \otimes A' \otimes I)U)$$

$$(B \otimes I \otimes I) \Leftrightarrow (U^{*}(I \otimes I \otimes B')U)$$

$$probe\ (B, A') \Leftarrow [U^{*}(I \otimes A' \otimes I)U, U^{*}(B \otimes I \otimes I)U] = 0;$$

$$probe\ (C, B') \Leftarrow [U^{*}(I \otimes I \otimes B')U, U^{*}(C \otimes I \otimes I)U] = 0;$$

Quantum state = $(P \otimes P_{p} \otimes P_{q})$

Horn clauses for incompatibles A,B, and C

Probe A followed by measurement A – same answer

Probe A, probe B followed by measurement A
Suppose C=A; Different answers

$$probe\ (B, A') \Leftarrow [U^*(I \otimes A' \otimes I)U, U^*(B \otimes I \otimes I)U] = 0;$$

$$probe\ (A, B') \Leftarrow [U^*(I \otimes I \otimes B')U, U^*(A \otimes I \otimes I)U] = 0;$$

Quantum state = $(P \otimes P_p \otimes P_q)$

Summary and Future Work

- Probabilistic logic programming language for quantum h/w
- Turing computable and constructive logic
- Supports infinite probability distributions
- Extension to squeezed and non-Gaussian states
- To express and verify properties of more complex protocols
- Enriching the language with types
- Hybrid classical-quantum theorem prover
- Formalizing (Category theory based) quantum measurements as part of theorem proving process

Thank you for your attention